Problem 4.76

The shift in the energy levels in Example 4.6 can be understood from classical electrodynamics. Consider the case where initially no current flows in the solenoid. Now imagine slowly increasing the current.

(a) Calculate (from classical electrodynamics) the emf produced by the changing flux and show that the rate at which work is done on the charge confined to the ring can be written

$$\frac{dW}{d\Phi} = -q\,\frac{\omega}{2\pi},$$

where ω is the angular velocity of the particle.

(b) Calculate the z component of the mechanical angular momentum,[77]

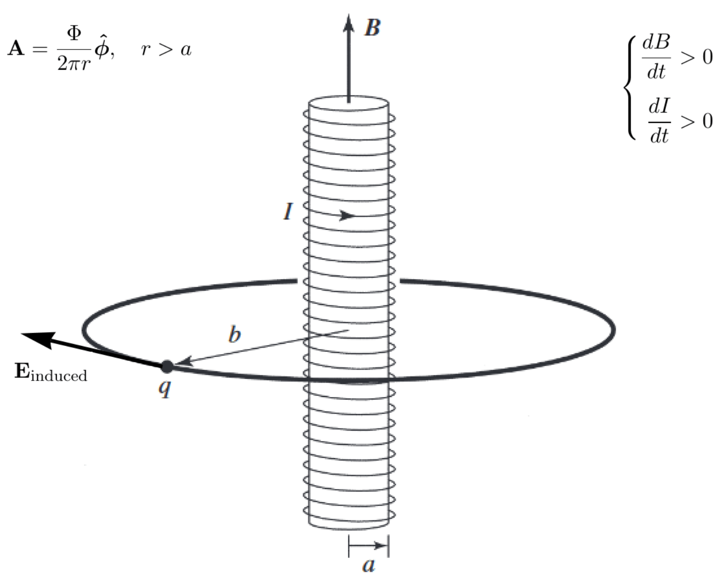
$$\mathbf{L}_{\text{mechanical}} = \mathbf{r} \times m\,\mathbf{v} = \mathbf{r} \times (\mathbf{p} - q\,\mathbf{A}), \tag{4.231}$$

for a particle in the state ψ_n in Example 4.6. Note that the *mechanical* angular momentum is not quantized in integer multiples of $\hbar$![78]

(c) Show that your result from part (a) is precisely equal to the rate at which the stationary state energies change as the flux is increased: $dE_n/d\Phi$.

Solution

Below is an illustration of the solenoid with a larger, coaxial ring of radius b. If the current increases, then the magnetic field within the solenoid becomes stronger. By Faraday's law, this induces an electric field within the ring to oppose the increase in magnetic flux.



[77]See footnote 62 for a discussion of the difference between the canonical and mechanical momentum.
[78]However, the electromagnetic fields *also* carry angular momentum, and the *total* (mechanical plus electromagnetic) is quantized in integer multiples of $\hbar$. For a discussion see M. Peshkin, *Physics Reports* **80**, 375 (1981) or Chapter 1 of Frank Wilczek, *Fractional Statistics and Anyon Superconductivity*, World Scientific, New Jersey (1990).

Part (a)

The induced electric field is tangent to the ring at every point and constant along the ring's circumference. According to Faraday's law,

$$\begin{aligned}
-\frac{d\Phi}{dt} &= \oint \mathbf{E}_{\text{induced}} \cdot d\mathbf{l} \\
&= E_{\text{induced}} \oint\limits_{x^2+y^2=b^2} dl \\
&= E_{\text{induced}} \cdot 2\pi b,
\end{aligned}$$

which means

$$E_{\text{induced}} = -\frac{1}{2\pi b}\frac{d\Phi}{dt}.$$

The work that this field does on the charge is

$$\begin{aligned}
dW &= \mathbf{F} \cdot d\mathbf{l} \\
&= q\mathbf{E}_{\text{induced}} \cdot d\mathbf{l} \\
&= q(E_{\text{induced}}\hat{\boldsymbol{\phi}}) \cdot (b\, d\phi\, \hat{\boldsymbol{\phi}}) \\
&= qbE_{\text{induced}}\, d\phi \\
&= qb\left(-\frac{1}{2\pi b}\frac{d\Phi}{dt}\right) d\phi \\
&= q\left(-\frac{1}{2\pi}\frac{d\Phi}{dt}\right) \omega\, dt \\
&= -q\frac{\omega}{2\pi}\, d\Phi.
\end{aligned}$$

Therefore,

$$\frac{dW}{d\Phi} = -q\frac{\omega}{2\pi}.$$

Part (b)

Start by rewriting the mechanical angular momentum.

$$\begin{aligned}
\mathbf{L}_{\text{mechanical}} &= \mathbf{r} \times m\mathbf{v} \\
&= \mathbf{r} \times (\mathbf{p} - q\mathbf{A}) \\
&= \mathbf{r} \times \mathbf{p} - q(\mathbf{r} \times \mathbf{A}) \\
&= \mathbf{L} - q\left[(r\hat{\mathbf{r}}) \times \left(\frac{\Phi}{2\pi r}\hat{\boldsymbol{\phi}}\right)\right] \\
&= \mathbf{L} - q\frac{\Phi}{2\pi}(\hat{\mathbf{r}} \times \hat{\boldsymbol{\phi}}) \\
&= \mathbf{L} - q\frac{\Phi}{2\pi}\hat{\mathbf{z}}
\end{aligned}$$

The z-component of the mechanical angular momentum is

$$\begin{aligned}
L_{\text{mechanical},z} &= L_z - q\frac{\Phi}{2\pi} \\
&= -i\hbar\frac{\partial}{\partial\phi} - q\frac{\Phi}{2\pi}.
\end{aligned}$$

Let it act on ψ_n to determine the eigenvalue.

$$\begin{aligned}
L_{\text{mechanical},z}\psi_n &= \left(-i\hbar\frac{\partial}{\partial\phi} - q\frac{\Phi}{2\pi}\right)\psi_n \\
&= \left(-i\hbar\frac{\partial}{\partial\phi} - q\frac{\Phi}{2\pi}\right)Ae^{in\phi} \\
&= -i\hbar\frac{d}{d\phi}(Ae^{in\phi}) - q\frac{\Phi}{2\pi}(Ae^{in\phi}) \\
&= -i\hbar(inAe^{in\phi}) - q\frac{\Phi}{2\pi}(Ae^{in\phi}) \\
&= n\hbar(Ae^{in\phi}) - q\frac{\Phi}{2\pi}(Ae^{in\phi}) \\
&= \left(n\hbar - q\frac{\Phi}{2\pi}\right)Ae^{in\phi} \\
&= \left(n\hbar - q\frac{\Phi}{2\pi}\right)\psi_n
\end{aligned}$$

Therefore, if you measure the z-component of the mechanical angular momentum for a particle in the state ψ_n, you will get

$$n\hbar - q\frac{\Phi}{2\pi}.$$

Part (c)

The stationary state energies were determined in Example 4.6. (See Equation 4.206 on page 184.)

$$E_n = \frac{\hbar^2}{2mb^2}\left(n - \frac{q\Phi}{2\pi\hbar}\right)^2, \quad (n = 0, \pm 1, \pm 2, \ldots)$$

Calculate the derivative with respect to Φ by using the chain rule.

$$\begin{aligned}
\frac{dE_n}{d\Phi} &= \frac{d}{d\Phi}\left[\frac{\hbar^2}{2mb^2}\left(n - \frac{q\Phi}{2\pi\hbar}\right)^2\right] \\
&= \frac{\hbar^2}{2mb^2}\cdot 2\left(n - \frac{q\Phi}{2\pi\hbar}\right)\frac{d}{d\Phi}\left(n - \frac{q\Phi}{2\pi\hbar}\right) \\
&= \frac{\hbar^2}{mb^2}\left(n - \frac{q\Phi}{2\pi\hbar}\right)\left(-\frac{q}{2\pi\hbar}\right) \\
&= \frac{1}{mb^2}\left(n\hbar - q\frac{\Phi}{2\pi}\right)\left(-\frac{q}{2\pi}\right)
\end{aligned}$$

The first quantity in parentheses is the z-component of the particle's mechanical angular momentum from part (b), which can also be written as $I\omega = mb^2\omega$. I is the particle's moment of inertia about the z-axis.

$$\begin{aligned}
\frac{dE_n}{d\Phi} &= \frac{1}{mb^2}(I\omega)\left(-\frac{q}{2\pi}\right) \\
&= \frac{1}{mb^2}(mb^2\omega)\left(-\frac{q}{2\pi}\right) \\
&= -q\frac{\omega}{2\pi} \\
&= \frac{dW}{d\Phi}
\end{aligned}$$

This is the result of part (a).